

# The non autonomous YdKN equation and generalized symmetries of Boll equations.

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## Abstract

In this paper we study the integrability of a class of nonlinear non autonomous quad graph equations compatible around the cube introduced by Boll. We show that all these equations possess three point generalized symmetries which are subcases of either the Yamilov discretization of the Krichever–Novikov equation or of its non autonomous extension. We also prove that all those symmetries are integrable as pass the algebraic entropy test.

## 1 Introduction

In 1983 Ravil I. Yamilov [20] classified all differential difference equations of the class  $\dot{u}_n = f(u_{n-1}, u_n, u_{n+1})$  using the generalized symmetry method. From the generalized symmetry method one obtains integrability conditions which allow to check whether a given equation is integrable. Moreover in many cases these conditions enable us to classify equations, i.e. to obtain complete lists of integrable equations belonging to a certain class. As integrability conditions are only necessary conditions for the existence of generalized symmetries and/or conservation laws, one then has to prove that the equations of the resulting list really possess generalized symmetries and conservation laws of sufficiently high order. One mainly construct them using Miura-type transformations and master symmetries, proving the existence of Lax pairs [21, 22]. The result of Yamilov classification, up to Miura transformation, is the Toda equation and the so called Yamilov discretization of the Krichever Novikov equation (YdKN), a differential difference equation depending on 6 arbitrary coefficients:

$$\frac{dq_k}{dt} = \frac{A(q_k)q_{k+1}q_{k-1} + B(q_k)(q_{k+1} + q_{k-1}) + C(q_k)}{q_{k+1} - q_{k-1}}, \quad (1)$$

where:

$$A(q_k) = aq_k^2 + 2bq_k + c, \quad (2a)$$

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$$B(q_k) = bq_k^2 + dq_k + e, \quad (2b)$$

$$C(q_k) = cq_k^2 + 2eq_k + f. \quad (2c)$$

The integrability of (1) is proven by the existence of point symmetries [13] and of a master symmetries [22] from which one is able to construct an infinite hierarchy of generalized symmetries. The problem of finding the Bäcklund transformation and Lax pair in the general case seems to be still open.

In [15] the authors constructed a set of five conditions necessary for the existence of generalized symmetries for a class of differential-difference equations depending only on nearest neighbouring interaction. They used the conditions to propose the integrability of the following non autonomous generalization of the YdKN:

$$\frac{dq_k}{dt} = \frac{A_k(q_k)q_{k+1}q_{k-1} + B_k(q_k)(q_{k+1} + q_{k-1}) + C_k(q_k)}{q_{k+1} - q_{k-1}}, \quad (3)$$

where the now  $k$ -dependent coefficients are given by:

$$A_k(q_k) = aq_k^2 + 2b_kq_k + c_k, \quad (4a)$$

$$B_k(q_k) = b_{k+1}q_k^2 + dq_k + e_{k+1}, \quad (4b)$$

$$C_k(q_k) = c_{k+1}q_k^2 + 2e_kq_k + f, \quad (4c)$$

with  $b_k$ ,  $c_k$  and  $e_k$  2-periodic functions. Eq. (3) has conservation laws of second and third order and two generalized local symmetries of order  $i$  and  $i + 1$ , with  $i < 4$ .

It was proved in [12] that the three point symmetries of the equations belonging to the so-called ABS classification [1], found systematically in [16], are all particular cases of (1). Here in this note we will show that the three point generalized symmetries of all the equations coming from the classification of Boll [2, 4–6], which extends the ABS one [1], are all particular cases of the YdKN or the non autonomous YdKN. In particular we will present the symmetries of all the classes of equations  $H^4$  and  $H^6$ , noting that the symmetries of the rhombic  $H^4$  were found firstly in [19]. For the remaining classes of equations namely the trapezoidal  $H^4$  and the  $H^6$  equations, which were found to be *linearizable* in [9], this is the first time that their generalized symmetries are presented. Furthermore we also present a new suggestion for the integrability of the non-autonomous YdKN (3) based on the algebraic entropy test and use the same criterion to prove the integrability of the other non autonomous equations of the  $H^4$  and  $H^6$  classes.

In Section 2 we present the three point generalized symmetries of the  $H^4$  and  $H^6$  classes and identify them with subcases of the YdKN or of its non autonomous extension. In Section 3 we compute the algebraic entropy for the non autonomous YdKN and its subcases obtained before while in Section 4 we present some brief conclusions.

## 2 Three point generalized symmetries and their identification

In this Section we consider the various classes of equation coming from the classification of Boll [2, 4–6] as presented in [9], show their symmetries and show

the identification of the fluxes of such symmetries with the YdKN and its non autonomous extension.

## 2.1 Rhombic $H^4$ equations

Once written on the  $\mathbb{Z}_{(n,m)}^2$  lattice, according to [19], the three equations belonging to this class have the form:

$${}_rH_1^\varepsilon: \quad (u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) - (\alpha - \beta) + \varepsilon(\alpha - \beta) \left( F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1} + F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} \right) = 0, \quad (5a)$$

$${}_rH_2^\varepsilon: \quad (u_{n,m} - u_{n+1,m+1})(u_{n+1,m} - u_{n,m+1}) + (\beta - \alpha)(u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) - \alpha^2 + \beta^2 - \varepsilon(\beta - \alpha)^3 - \varepsilon(\beta - \alpha) \left( 2F_{n+m}^{(-)} u_{n,m} + 2F_{n+m}^{(+)} u_{n+1,m} + \alpha + \beta \right) \cdot \left( 2F_{n+m}^{(-)} u_{n+1,m+1} + 2F_{n+m}^{(+)} u_{n,m+1} + \alpha + \beta \right) = 0, \quad (5b)$$

$${}_rH_3^\varepsilon: \quad \alpha(u_{n,m} u_{n+1,m} + u_{n,m+1} u_{n+1,m+1}) - \beta(u_{n,m} u_{n,m+1} + u_{n+1,m} u_{n+1,m+1}) + (\alpha^2 - \beta^2)\delta - \frac{\varepsilon(\alpha^2 - \beta^2)}{\alpha\beta} \left( F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1} + F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} \right) = 0, \quad (5c)$$

where

$$F_k^\pm = \frac{1 \pm (-1)^k}{2}, \quad k \in \mathbb{Z}. \quad (6)$$

Their generalized symmetries [19] are given by:

$$\widehat{X}_n^{r,H_1^\varepsilon} = \frac{1 - \varepsilon \left( F_{n+m}^{(+)} u_{n+1,m} u_{n-1,m} + F_{n+m}^{(-)} u_{n,m}^2 \right)}{u_{n+1,m} - u_{n-1,m}} \partial_{u_{n,m}}, \quad (7a)$$

$$\widehat{X}_m^{r,H_1^\varepsilon} = \frac{1 - \varepsilon \left( F_{n+m}^{(+)} u_{n,m+1} u_{n,m-1} + F_{n+m}^{(-)} u_{n,m}^2 \right)}{u_{n,m+1} - u_{n,m-1}} \partial_{u_{n,m}}, \quad (7b)$$

$$\widehat{X}_n^{r,H_2^\varepsilon} = \left[ \frac{\left( 1 - 4\varepsilon\alpha F_{n+m}^{(-)} \right) (u_{n+1,m} + u_{n-1,m}) - 4\varepsilon F_{n+m}^{(+)} u_{n+1,m} u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} + \frac{2\alpha - 4\varepsilon\alpha^2 - 4\varepsilon F_{n+m}^{(-)} u_{n,m}^2 + \left( 1 - 4\varepsilon\alpha F_{n+m}^{(-)} \right) u_{n,m}}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}} \quad (7c)$$

$$\widehat{X}_m^{r,H_2^\varepsilon} = \left[ \frac{\left( 1 - 4\varepsilon\beta F_{n+m}^{(-)} \right) (u_{n,m+1} + u_{n,m-1}) - 4\varepsilon F_{n+m}^{(+)} u_{n,m+1} u_{n,m-1}}{u_{n,m+1} - u_{n,m-1}} + \frac{2\beta - 4\varepsilon\beta^2 - 4\varepsilon F_{n+m}^{(-)} u_{n,m}^2 + \left( 1 - 4\varepsilon\beta F_{n+m}^{(-)} \right) u_{n,m}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}} \quad (7d)$$

Eq.	$k$	$a$	$b_k$	$c_k$	$d$	$e_k$	$f$
${}_r H_1^\varepsilon$	$n$	0	0	$-\varepsilon F_{n+m}^{(+)}$	0	0	1
	$m$	0	0	$-\varepsilon F_{n+m}^{(+)}$	0	0	1
${}_r H_2^\varepsilon$	$n$	0	0	$-4\varepsilon F_{n+m}^{(+)}$	0	$1 - 4\varepsilon\alpha F_{n+m}^{(-)}$	$2\alpha - 4\varepsilon\alpha^2$
	$m$	0	0	$-4\varepsilon F_{n+m}^{(+)}$	0	$1 - 4\varepsilon\beta F_{n+m}^{(-)}$	$2\beta - 4\varepsilon\beta^2$
${}_r H_3^\varepsilon$	$n$	0	0	$-\frac{\varepsilon F_{n+m}^{(+)}}{\alpha}$	$\frac{1}{2}$	0	$\delta\alpha$
	$m$	0	0	$-\frac{\varepsilon F_{n+m}^{(+)}}{\beta}$	$\frac{1}{2}$	0	$\delta\beta$

Table 1: Identification of the coefficients in the symmetries of the rhombic  $H^4$  equations with those of the non autonomous YdKN equation.

$$\widehat{X}_n^{rH_3^\varepsilon} = \left[ \frac{1}{2} \frac{u_{n,m}(u_{n+1,m} + u_{n-1,m}) + 2\delta\alpha}{u_{n+1,m} - u_{n-1,m}} - \frac{\varepsilon}{\alpha} \frac{(F_{n+m}^{(+)} u_{n+1,m} u_{n-1,m} + F_{n+m}^{(-)} u_{n,m}^2)}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}} \quad (7e)$$

$$\widehat{X}_m^{rH_3^\varepsilon} = \left[ \frac{1}{2} \frac{u_{n,m}(u_{n,m+1} + u_{n,m-1}) + 2\delta\beta}{u_{n,m+1} + u_{n,m-1}} - \frac{\varepsilon}{\beta} \frac{(F_{n+m}^{(+)} u_{n,m+1} u_{n,m-1} + F_{n+m}^{(-)} u_{n,m}^2)}{u_{n,m+1} + u_{n,m-1}} \right] \partial_{u_{n,m}} \quad (7f)$$

As stated in [19] the fluxes of the symmetries (7) are readily identified with the corresponding cases of the non autonomous YdKN equation (3). Such identification is in this paper made explicit by showing the appropriate values of the coefficients of (3) in Table 1.

## 2.2 Trapezoidal $H^4$ equations

We now consider the trapezoidal  $H^4$  equations, which appeared in [5, 6] and whose non-autonomous form was given in [9]:

$${}_t H_1: (u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) - \alpha_2 \varepsilon^2 \left( F_m^{(+)} u_{n,m+1} u_{n+1,m+1} + F_m^{(-)} u_{n,m} u_{n+1,m} \right) - \alpha_2 = 0, \quad (8a)$$

$$\begin{aligned} {}_t H_2: & (u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) \\ & + \alpha_2 (u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) \\ & + \frac{\varepsilon\alpha_2}{2} \left( 2F_m^{(+)} u_{n,m+1} + 2\alpha_3 + \alpha_2 \right) \left( 2F_m^{(+)} u_{n+1,m+1} + 2\alpha_3 + \alpha_2 \right) \\ & + \frac{\varepsilon\alpha_2}{2} \left( 2F_m^{(-)} u_{n,m} + 2\alpha_3 + \alpha_2 \right) \left( 2F_m^{(-)} u_{n+1,m} + 2\alpha_3 + \alpha_2 \right) \\ & + (\alpha_3 + \alpha_2)^2 - \alpha_3^2 - 2\varepsilon\alpha_2\alpha_3(\alpha_3 + \alpha_2) = 0 \end{aligned} \quad (8b)$$

$$\begin{aligned}
{}_tH_3: & \alpha_2 (u_{n,m}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1}) \\
& - (u_{n,m}u_{n,m+1} + u_{n+1,m}u_{n+1,m+1}) - \alpha_3 (\alpha_2^2 - 1) \delta^2 + \\
& - \frac{\varepsilon^2(\alpha_2^2 - 1)}{\alpha_3\alpha_2} \left( F_m^{(+)}u_{n,m+1}u_{n+1,m+1} + F_m^{(-)}u_{n,m}u_{n+1,m} \right) = 0,
\end{aligned} \tag{8c}$$

We can easily calculate the three points generalized symmetries of  ${}_tH_2^\varepsilon$  (8b) and of  ${}_tH_3^\varepsilon$  (8c):

$$\begin{aligned}
\widehat{X}_n^{tH_2^\varepsilon} = & \left[ \frac{(u_{n,m} + \varepsilon\alpha_2^2 F_m^{(+)})(u_{n+1,m} + u_{n-1,m}) - u_{n+1,m}u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} - \right. \\
& \left. - \frac{u_{n,m}^2 - 2\varepsilon F_m^{(+)}\alpha_2^2 u_{n,m} - \alpha_2^2 + 4\varepsilon F_m^{(+)}\alpha_3^2 + 8\varepsilon F_m^{(+)}\alpha_2^2\alpha_3 + \varepsilon^2 F_m^{(+)}\alpha_2^4}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}},
\end{aligned} \tag{9a}$$

$$\begin{aligned}
\widehat{X}_m^{tH_2^\varepsilon} = & \left[ \frac{\left[ \frac{1}{2} - \varepsilon(\alpha_2 + \alpha_3)F_m^{(+)} \right] (u_{n,m+1} + u_{n,m-1}) - \varepsilon F_m^{(+)}u_{n,m+1}u_{n,m-1}}{u_{n,m+1} - u_{n,m-1}} - \right. \\
& \left. - \frac{\varepsilon F_m^{(-)}u_{n,m}^2 - \left[ 1 - 2\varepsilon(\alpha_2 + \alpha_3)F_m^{(-)} \right] u_{n,m} + \alpha_3 + \varepsilon(\alpha_2 + \alpha_3)^2}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}},
\end{aligned} \tag{9b}$$

$$\begin{aligned}
\widehat{X}_n^{tH_3^\varepsilon} = & \left[ \frac{\frac{1}{2}\alpha_2(1 + \alpha_2^2)u_{n,m}(u_{n+1,m} + u_{n-1,m}) - \alpha_2^2 u_{n+1,m}u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} - \right. \\
& \left. - \frac{\alpha_2^2 u_{n,m}^2 + \varepsilon^2 \delta^2 (1 - \alpha_2^2)^2 F_m^{(+)}}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}},
\end{aligned} \tag{9c}$$

$$\begin{aligned}
\widehat{X}_m^{tH_3^\varepsilon} = & \left[ \frac{\frac{1}{2}\alpha_3 u_{n,m}(u_{n,m+1} + u_{n,m-1}) - \varepsilon^2 F_m^{(+)}u_{n,m+1}u_{n,m-1}}{u_{n,m+1} - u_{n,m-1}} - \right. \\
& \left. - \frac{\varepsilon^2 F_m^{(-)}u_{n,m}^2 + \alpha_3^2 \delta^2}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}},
\end{aligned} \tag{9d}$$

The symmetries in the  $n$  and  $m$  directions and the linearizations of the  ${}_tH_1^\varepsilon$  equation (8a) have been presented in [10]. Their peculiarity is that they are determined by two arbitrary functions of one continuous variable and one lattice index and by arbitrary functions of the lattice indices. This is the first time that we find a lattice equation whose generalized symmetries depend on arbitrary functions. Almost surely this peculiarity is related to the very specific way in which  ${}_tH_1^\varepsilon$  is linearizable. Here we present only the sub-cases which are related to the YdKN equation in its autonomous or non autonomous form.

The general symmetry in the  $n$  direction is:

$$\begin{aligned}
\hat{X}_n^{tH_1^\varepsilon} = & F_m^{(+)} \left\{ \frac{\alpha_2 (v^2 + \varepsilon^2 \alpha_2^2)}{(r-v)(r+v)} B_n \left( \frac{\alpha_2}{r} \right) - \frac{\alpha_2 (r^2 + \varepsilon^2 \alpha_2^2)}{(r-v)(r+v)} B_{n-1} \left( \frac{\alpha_2}{v} \right) + \right. \\
& + \left[ u_{n,m} - \frac{(r^2 + \varepsilon^2 \alpha_2^2) v}{(r-v)(r+v)} \right] \alpha + \gamma_m \left. \right\} \partial_{u_{n,m}} + F_m^{(-)} \left[ \frac{s^2 t^2}{(s-t)(s+t)} (B_n(s) - \right. \\
& - B_{n-1}(t)) - \frac{s^2 t}{(s-t)(s+t)} \alpha + \delta_m \left. \right] \left( 1 + \varepsilon^2 u_{n,m}^2 \right) \partial_{u_{n,m}}, \quad r \doteq u_{n+1,m} - u_{n,m}, \\
s \doteq & \frac{u_{n+1,m} - u_{n,m}}{1 + \varepsilon^2 u_{n+1,m} u_{n,m}}, \quad t \doteq \frac{u_{n,m} - u_{n-1,m}}{1 + \varepsilon^2 u_{n-1,m} u_{n,m}}, \quad v \doteq u_{n,m} - u_{n-1,m},
\end{aligned} \tag{10}$$

where  $B_n(x)$ ,  $\gamma_m$  and  $\delta_m$  are generic functions of their arguments and  $\alpha$  is an arbitrary parameter. When  $B_n(x) = -1/x$ ,  $\alpha = \gamma_m = \delta_m = 0$ , we get

$$\hat{X}_n^{tH_1^\varepsilon} = \left[ \frac{(u_{n+1,m} - u_{n,m})(u_{n,m} - u_{n-1,m})}{u_{n+1,m} - u_{n-1,m}} - F_m^{(+)} \frac{\varepsilon^2 \alpha_2^2}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}}. \tag{11}$$

The general symmetry in the  $m$  direction is:

$$\begin{aligned}
\hat{X}_m^{tH_1^\varepsilon} = & [F_m^{(+)} \left( B_m \left( \frac{u_{n,m+1} - u_{n,m-1}}{1 + \varepsilon^2 u_{n,m+1} u_{n,m-1}} \right) + \kappa_m \right) \\
& + F_m^{(-)} (1 + \varepsilon^2 u_{n,m}^2) (C_m(u_{n,m+1} - u_{n,m-1}) + \lambda_m)] \partial_{u_{n,m}}.
\end{aligned} \tag{12}$$

When  $B_m(t) = 1/t$ ,  $C_m(t) = 1/t$  and  $\kappa_m = \lambda_m = 0$  (12) becomes

$$\hat{X}_m^{tH_1^\varepsilon} = [F_m^{(+)} \frac{1 + \varepsilon^2 u_{n,m+1} u_{n,m-1}}{u_{n,m+1} - u_{n,m-1}} + F_m^{(-)} \frac{1 + \varepsilon^2 u_{n,m}^2}{u_{n,m+1} - u_{n,m-1}}] \partial_{u_{n,m}}. \tag{13}$$

Let us notice that the symmetries (9, 11) in the  $n$  direction are sub-cases of the original YdKN equation. As  $F_m^{(\pm)}$  depends on the other lattice index, it can be treated like a parameter which is either 0 or 1.

The explicit identification of the coefficients of the symmetries (9), (11) and (13) is shown in Table 2.

### 2.3 $H^6$ equations

In this subsection we consider the equations of the family  $H^6$  introduced in [5,6]. We shall present their non autonomous form on the lattice  $\mathbb{Z}_{(n,m)}^2$  as given in [9]:

$$\begin{aligned}
{}_1D_2: \quad & \left( F_{n+m}^{(-)} - \delta_1 F_n^{(+)} F_m^{(-)} + \delta_2 F_n^{(+)} F_m^{(+)} \right) u_{n,m} \\
& + \left( F_{n+m}^{(+)} - \delta_1 F_n^{(-)} F_m^{(-)} + \delta_2 F_n^{(-)} F_m^{(+)} \right) u_{n+1,m} + \\
& + \left( F_{n+m}^{(+)} - \delta_1 F_n^{(+)} F_m^{(+)} + \delta_2 F_n^{(+)} F_m^{(-)} \right) u_{n,m+1} \\
& + \left( F_{n+m}^{(-)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(-)} \right) u_{n+1,m+1} + \\
& + \delta_1 \left( F_m^{(-)} u_{n,m} u_{n+1,m} + F_m^{(+)} u_{n,m+1} u_{n+1,m+1} \right) \\
& + F_{n+m}^{(+)} u_{n,m} u_{n+1,m+1} + F_{n+m}^{(-)} u_{n+1,m} u_{n,m+1} = 0,
\end{aligned} \tag{14a}$$

Eq.	$k$	$a$	$b_k$	$c_k$	$d$	$e_k$	$f$
${}_tH_1^\varepsilon$	$n$	0	0	-1	1	0	$-\varepsilon^2 \alpha_2^2 F_m^{(+)}$
	$m$	0	0	$\varepsilon^2 F_m^{(+)}$	0	0	2
${}_tH_2^\varepsilon$	$n$	0	0	-1	1	$\varepsilon \alpha_2^2 F_m^{(+)}$	$\alpha_2^2 - \varepsilon \alpha_2^2 (4\alpha_2 + 8\alpha_3 + \varepsilon \alpha_2^2) F_m^{(+)}$
	$m$	0	0	$-\varepsilon F_m^{(+)}$	0	$\frac{1}{2} - \varepsilon(\alpha_2 + \alpha_3) F_m^{(-)}$	$-\alpha_3 - \varepsilon(\alpha_2 + \alpha_3)^2$
${}_tH_3^\varepsilon$	$n$	0	0	$-\alpha_2^2$	$\frac{1}{2} \alpha_2 (1 + \alpha_2^2)$	0	$-\varepsilon^2 \delta^2 F_m^{(+)} (1 - \alpha_2^2)^2$
	$m$	0	0	$-\varepsilon^2 F_m^{(+)}$	$\frac{1}{2} \alpha_3$	0	$-\alpha_3^2 \delta^2$

Table 2: Identification of the coefficients in the symmetries of the trapezoidal  $H^4$  equations with those of the YdKN equation. In the direction  $n$  the YdKN is autonomous while in the  $m$  direction is non autonomous. Here the symmetries of  ${}_tH_1^\varepsilon$  in the  $m$  direction are the subcase (13) of (12) while those in the  $n$  direction are the subcase (11) of (10).

$${}_2D_2: \quad \left( F_m^{(-)} - \delta_1 F_n^{(+)} F_m^{(-)} + \delta_2 F_n^{(+)} F_m^{(+)} - \delta_1 \lambda F_n^{(-)} F_m^{(+)} \right) u_{n,m} \quad (14b)$$

$$+ \left( F_m^{(-)} - \delta_1 F_n^{(-)} F_m^{(-)} + \delta_2 F_n^{(-)} F_m^{(+)} - \delta_1 \lambda F_n^{(+)} F_m^{(+)} \right) u_{n+1,m} \\ + \left( F_m^{(+)} - \delta_1 F_n^{(+)} F_m^{(+)} + \delta_2 F_n^{(+)} F_m^{(-)} - \delta_1 \lambda F_n^{(-)} F_m^{(-)} \right) u_{n,m+1} \\ + \left( F_m^{(+)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(-)} - \delta_1 \lambda F_n^{(+)} F_m^{(-)} \right) u_{n+1,m+1} \\ + \delta_1 \left( F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} + F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1} \right) \\ + F_m^{(+)} u_{n,m} u_{n+1,m} + F_m^{(-)} u_{n,m+1} u_{n+1,m+1} - \delta_1 \delta_2 \lambda = 0,$$

$${}_3D_2: \quad \left( F_m^{(-)} - \delta_1 F_n^{(-)} F_m^{(-)} + \delta_2 F_n^{(+)} F_m^{(+)} - \delta_1 \lambda F_n^{(-)} F_m^{(+)} \right) u_{n,m} \quad (14c)$$

$$+ \left( F_m^{(-)} - \delta_1 F_n^{(+)} F_m^{(-)} + \delta_2 F_n^{(-)} F_m^{(+)} - \delta_1 \lambda F_n^{(+)} F_m^{(+)} \right) u_{n+1,m} \\ + \left( F_m^{(+)} - \delta_1 F_n^{(-)} F_m^{(+)} + \delta_2 F_n^{(+)} F_m^{(-)} - \delta_1 \lambda F_n^{(-)} F_m^{(-)} \right) u_{n,m+1} \\ + \left( F_m^{(+)} - \delta_1 F_n^{(+)} F_m^{(+)} + \delta_2 F_n^{(-)} F_m^{(-)} - \delta_1 \lambda F_n^{(+)} F_m^{(-)} \right) u_{n+1,m+1} \\ + \delta_1 \left( F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1} \right) \\ + F_m^{(-)} u_{n,m+1} u_{n+1,m+1} + F_m^{(+)} u_{n,m} u_{n+1,m} - \delta_1 \delta_2 \lambda = 0,$$

$$D_3: \quad F_n^{(+)} F_m^{(+)} u_{n,m} + F_n^{(-)} F_m^{(+)} u_{n+1,m} + F_n^{(+)} F_m^{(-)} u_{n,m+1} \quad (14d)$$

$$+ F_n^{(-)} F_m^{(-)} u_{n+1,m+1} + F_m^{(-)} u_{n,m} u_{n+1,m} \\ + F_n^{(-)} u_{n,m} u_{n,m+1} + F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} + \\ + F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1} \\ + F_m^{(+)} u_{n,m+1} u_{n+1,m+1} = 0,$$

$${}_1D_4: \quad \delta_1 \left( F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1} \right) + \quad (14e)$$

$$+ \delta_2 \left( F_m^{(-)} u_{n,m} u_{n+1,m} + F_m^{(+)} u_{n,m+1} u_{n+1,m+1} \right) + \\ + u_{n,m} u_{n+1,m+1} + u_{n+1,m} u_{n,m+1} + \delta_3 = 0,$$

$${}_2D_4: \quad \delta_1 \left( F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1} \right) + \quad (14f)$$

$$+ \delta_2 \left( F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} + F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1} \right) + \\ + u_{n,m} u_{n+1,m} + u_{n,m+1} u_{n+1,m+1} + \delta_3 = 0.$$

The three forms of the equation  $D_2$  (14a,14b,14c), which we will collectively call  ${}_iD_2$  assuming  $i$  in  $\{1, 2, 3\}$ , possess the following three points generalized symmetries in the  $n$  direction and three points generalized symmetries in the  $m$



direction:

$$\begin{aligned} \hat{X}_n^{1D_2} = & \left[ \frac{\left( F_n^{(+)} F_m^{(-)} - \delta_1 F_n^{(+)} F_m^{(+)} \right) (u_{n,m+1} + u_{n-1,m})}{u_{n+1,m} - u_{n-1,m}} + \right. \\ & + \frac{\left( F_n^{(+)} F_m^{(-)} - \delta_1 F_n^{(+)} F_m^{(+)} - \delta_1 \delta_2 \right) F_n^{(-)} F_m^{(+)} u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} + \\ & \left. + \frac{(F_{n+m}^{(+)} - \delta_1 F_m^{(+)} - \delta_1 \delta_2 F_n^{(+)} F_m^{(+)} u_{n,m} + \delta_2 F_m^{(-)})}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}}, \end{aligned} \quad (15a)$$

$$\begin{aligned} \hat{X}_m^{1D_2} = & \left[ \frac{\delta_1 F_n^{(-)} F_m^{(+)} u_{n,m+1} u_{n,m-1} + F_{n+m}^{(-)} (u_{n,m+1} + u_{n,m-1})}{u_{n,m+1} - u_{n,m-1}} + \right. \\ & + \frac{\delta_1 F_m^{(+)} u_{n,m+1} + \delta_1 \delta_2 F_n^{(-)} F_m^{(+)} u_{n,m-1} + \delta_1 F_n^{(-)} F_m^{(-)} u_{n,m}^2}{u_{n,m+1} - u_{n,m-1}} + \\ & + \frac{\left[ F_{n+m}^{(+)} + \delta_1 \left( F_n^{(+)} F_m^{(-)} - F_n^{(-)} F_m^{(-)} \right) + \delta_1 \delta_2 F_n^{(-)} F_m^{(-)} \right] u_{n,m}}{u_{n,m+1} - u_{n,m-1}} - \\ & \left. - \frac{\delta_2 (\delta_1 - 1) F_n^{(-)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{nm}}, \end{aligned} \quad (15b)$$

$$\begin{aligned} \hat{X}_n^{2D_2} = & \left[ \frac{\left( F_n^{(-)} F_m^{(-)} \delta_1 + F_n^{(-)} F_m^{(-)} \delta_1 \delta_2 - F_n^{(-)} F_m^{(-)} \right) u_{n+1,m} + \left( F_n^{(+)} F_m^{(+)} \delta_1 - F_n^{(+)} F_m^{(-)} \right) u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} + \right. \\ & + \frac{\left( \delta_1 F_{n+m}^{(-)} - F_m^{(-)} + \delta_1 \delta_2 F_n^{(+)} F_m^{(-)} \right) u_{n,m} - (\delta_1 - 1) F_m^{(+)}}{u_{n+1,m} - u_{n-1,m}} \left. \right] \partial_{u_{n,m}}, \end{aligned} \quad (15c)$$

$$\begin{aligned} \hat{X}_m^{2D_2} = & \left[ \frac{F_n^{(-)} F_m^{(-)} \delta_1 u_{n,m+1} u_{n,m-1} + \left( \delta_1 \delta_2 F_n^{(-)} F_m^{(-)} + F_n^{(+)} F_m^{(-)} \right) u_{n,m+1}}{u_{n,m+1} - u_{n,m-1}} + \right. \\ & + \frac{\left( \delta_1 F_n^{(+)} F_m^{(+)} + F_n^{(-)} F_m^{(-)} - \delta_1 F_n^{(-)} F_m^{(-)} \right) u_{n,m-1}}{u_{n,m+1} - u_{n,m-1}} + \\ & + \frac{\delta_1 F_n^{(-)} F_m^{(+)} u_{n,m}^2 + \left[ F_n^{(+)} F_m^{(-)} + (\delta_2 - 1) F_n^{(-)} F_m^{(+)} + F_m^{(+)} \right] u_{n,m}}{u_{n,m+1} - u_{n,m-1}} + \\ & \left. + \frac{\delta_2 (1 - \delta_2) F_n^{(-)} - \delta_1 \lambda F_n^{(+)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}}, \end{aligned} \quad (15d)$$

$$\begin{aligned} \hat{X}_n^{3D_2} = & \left[ \frac{\left( \delta_1 F_n^{(+)} F_m^{(-)} + \delta_1 \delta_2 F_n^{(+)} F_m^{(-)} - F_n^{(+)} F_m^{(-)} \right) u_{n+1,m}}{u_{n+1,m} - u_{n-1,m}} + \right. \\ & + \frac{\left( F_n^{(+)} F_m^{(+)} \delta_1 - F_n^{(-)} F_m^{(-)} \right) u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} + \\ & \left. + \frac{\left( \delta_1 F_n^{(-)} F_m^{(-)} \delta_2 + F_n^{(-)} \delta_1 - F_m^{(-)} \right) u_{n,m} + (1 - \delta_1) F_m^{(+)}}{u_{n+1,m} - u_{n-1,m}} \right] \partial u_{n,m}, \end{aligned} \quad (15e)$$

$$\begin{aligned} \hat{X}_m^{3D_2} = & \left[ \frac{(1 - \delta_1 - \delta_1 \delta_2) F_n^{(+)} F_m^{(-)} u_{n,m+1}}{u_{n,m+1} - u_{n,m-1}} + \right. \\ & + \frac{\left( F_n^{(-)} F_m^{(-)} - F_n^{(+)} F_m^{(+)} \delta_1 \right) u_{n,m-1} + \delta_2 F_n^{(-)}}{u_{n,m+1} - u_{n,m-1}} + \\ & + \frac{\left( F_m^{(+)} - \delta_1 F_n^{(+)} - \delta_1 \delta_2 F_n^{(+)} F_m^{(+)} \right) u_{n,m}}{u_{n,m+1} - u_{n,m-1}} - \\ & \left. - \frac{\lambda \delta_1 (1 - \delta_1 - \delta_1 \delta_2) F_n^{(+)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{unm}. \end{aligned} \quad (15f)$$

It can be readily proved that these symmetries are not non autonomous YdKN equations (3), however the equations  ${}_i D_2$  possess also the following point symmetries:

$$\hat{Y}_1^{1D_2} = \left( F_n^{(+)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} + F_n^{(-)} F_m^{(+)} \right) u_{n,m} \partial_{u_{n,m}}, \quad (16a)$$

$$\hat{Y}_2^{1D_2} = \left[ \delta_1 F_n^{(+)} F_m^{(+)} + [1 - \delta_1 (1 + \delta_2)] F_n^{(-)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} \right] \partial_{u_{n,m}}, \quad (16b)$$

$$\begin{aligned} \hat{Y}_1^{2D_2} = & \left[ \left( F_n^{(+)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} + F_n^{(-)} F_m^{(+)} \right) u_{n,m} - \right. \\ & \left. - \lambda F_n^{(+)} F_m^{(-)} + \lambda [1 - \delta_1 (1 + \delta_2)] F_n^{(-)} F_m^{(-)} \right] \partial_{u_{n,m}}, \end{aligned} \quad (16c)$$

$$\hat{Y}_2^{2D_2} = \left[ \delta_1 F_n^{(+)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} [1 - \delta_1 (1 + \delta_2)] F_n^{(-)} F_m^{(-)} \right] \partial_{u_{n,m}}, \quad (16d)$$

$$\begin{aligned} \hat{Y}_1^{3D_2} = & \left[ \left( F_n^{(+)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} + F_n^{(-)} F_m^{(+)} \right) u_{n,m} - \right. \\ & \left. - \lambda F_n^{(-)} F_m^{(-)} + \lambda [1 - \delta_1 (1 + \delta_2)] F_n^{(-)} F_m^{(-)} \right] \partial_{u_{n,m}}, \end{aligned} \quad (16e)$$

$$\hat{Y}_2^{3D_2} = \left[ \delta_1 F_n^{(+)} F_m^{(+)} + [1 - \delta_1 (1 + \delta_2)] F_n^{(+)} F_m^{(-)} - F_n^{(-)} F_m^{(-)} \right] \partial_{u_{n,m}}, \quad (16f)$$

As the symmetries (15) are not in the form of the YdKN equation (3), we may look for a linear combination:

$$\hat{Z}_j^{iD_2} = \hat{X}_j^{iD_2} + K_1 \hat{Y}_1^{iD_2} + K_2 \hat{Y}_2^{iD_2}, \quad j = n, m; \quad i = 1, 2, 3 \quad (17)$$

such that the resulting symmetries of equations  ${}_i D_2$  will be in the form (3). Indeed it turns out that this is the case and the resulting identification with the proper constants  $K_1$  and  $K_2$  is displayed in Table 3. The fact that the  ${}_i D_2$  equations admit point symmetries and generalized symmetries makes them a unique case among the equations of Boll classification.

Eq.	$k$	$a$	$b_k$	$c_k$	$d$	$e_k$	$f$	$K_1$	$K_2$
${}_1D_2$	$n$	0	0	0	0	$\frac{1}{2}[\delta_1(1+\delta_2)-1]F_n^{(+)}F_m^{(+)} + \frac{1}{2}F_n^{(-)}F_m^{(+)} - \frac{1}{2}F_n^{(-)}F_m^{(-)}$	$-\delta_2 F_m^{(-)}$	0	-1/2
	$m$	0	0	$-F_n^{(-)}F_m^{(+)}\delta_1$	0	$\frac{1}{2}(\delta_1(1-\delta_2)-1)F_n^{(-)}F_m^{(-)} - \frac{1}{2}F_n^{(+)}F_m^{(+)} - \frac{1}{2}\delta_1 F_n^{(-)}F_m^{(+)}$	$\delta_2(\delta_1-1)F_n^{(-)}$	0	-1/2
${}_2D_2$	$n$	0	0	0	0	$\frac{1}{2}[1-\delta_1(1+\delta_2)]F_n^{(+)}F_m^{(-)} + \frac{1}{2}F_n^{(-)}F_m^{(-)} - \frac{1}{2}\delta_1 F_n^{(-)}F_m^{(+)}$	$(\delta_1-1)F_m^{(+)}$	0	-1/2
	$m$	0	0	$-\delta_1 F_n^{(-)}F_m^{(-)}$	0	$\frac{1}{2}[\delta_1(1-\delta_2)-1]F_n^{(-)}F_m^{(+)} - \frac{1}{2}F_n^{(+)}F_m^{(+)} - \frac{1}{2}\delta_1 F_n^{(-)}F_m^{(+)}$	$\delta_2[\delta_1-1]F_n^{(-)} + \lambda\delta_1 F_n^{(+)}$	0	-1/2
${}_3D_2$	$n$	0	0	0	0	$\frac{1}{2}[\delta_1(1+\delta_2)-1]F_n^{(-)}F_m^{(-)} + \frac{1}{2}F_n^{(+)}F_m^{(-)} + \frac{1}{2}\delta_1 F_n^{(-)}F_m^{(+)}$	$(1-\delta_1)F_m^{(+)}$	0	1/2
	$m$	0	0	0	0	$\frac{1}{2}[\delta_1(1-\delta_2)-1]F_n^{(+)}F_m^{(+)} - \frac{1}{2}F_n^{(+)}F_m^{(-)} + \frac{1}{2}\delta_1 F_n^{(-)}F_m^{(+)}$	$\delta_1\lambda[-\delta_1(1+\delta_2)]F_n^{(+)} - \delta_2 F_n^{(-)}$	0	1/2

Table 3: Identification of the coefficients of the symmetries of the  ${}_iD_2$  equations and value of the constants  $K_1$  and  $K_2$  in (17) in order to obtain non autonomous YdKN equations.

The  $D_3$  equation (14d) admits only the following three points generalized symmetries<sup>1</sup>:

$$\hat{X}_n^{D_3} = \left[ \frac{F_n^{(+)} F_m^{(+)} u_{n+1,m} u_{n-1,m} + \frac{1}{2} (F_m^{(-)} - F_n^{(-)} F_m^{(+)}) u_{n,m} (u_{n+1,m} + u_{n-1,m})}{u_{n+1,m} - u_{n-1,m}} + \right. \\ \left. + \frac{F_n^{(-)} F_m^{(+)} u_{n,m}^2 + (F_m^{(-)} - F_n^{(+)} F_m^{(+)}) u_{n,m}}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}}, \quad (18a)$$

$$\hat{X}_m^{D_3} = \left[ \frac{F_n^{(+)} F_m^{(+)} u_{n,m+1} u_{n,m-1} + \frac{1}{2} (F_n^{(-)} - F_n^{(+)} F_m^{(-)}) u_{n,m} (u_{n,m+1} + u_{n,m-1})}{u_{n,m+1} - u_{n,m-1}} + \right. \\ \left. + \frac{F_n^{(+)} F_m^{(-)} u_{n,m}^2 + (F_n^{(-)} - F_n^{(+)} F_m^{(+)}) u_{n,m}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}} \quad (18b)$$

and no point symmetries. Also the two forms of  $D_4$  possess only the following three point generalized symmetries:

$$\hat{X}_n^{D_4} = \left[ \frac{-\delta_1 F_n^{(+)} u_{n+1,m} u_{n-1,m} - \frac{1}{2} u_{n,m} (u_{n+1,m} + u_{n-1,m})}{u_{n+1,m} - u_{n-1,m}} + \right. \\ \left. + \frac{-\delta_1 F_n^{(-)} u_{n,m}^2 + \delta_2 \delta_3 F_m^{(+)}}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}}, \quad (18c)$$

$$\hat{X}_m^{D_4} = \left[ \frac{F_m^{(-)} u_{n,m+1} u_{n,m-1} + \frac{1}{2} u_{n,m} (u_{n,m+1} + u_{n,m-1})}{u_{n,m+1} - u_{n,m-1}} + \right. \\ \left. + \frac{\delta_2 F_m^{(+)} u_{n,m}^2 - \delta_1 \delta_3 F_n^{(+)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}}, \quad (18d)$$

$$\hat{X}_n^{D_4} = \left[ \frac{-\delta_1 \delta_2 F_n^{(+)} F_m^{(+)} u_{n+1,m} u_{n-1,m} + \frac{1}{2} u_{n,m} (u_{n+1,m} + u_{n-1,m})}{u_{n+1,m} - u_{n-1,m}} + \right. \\ \left. + \frac{-\delta_1 \delta_2 F_n^{(-)} F_m^{(+)} u_{n,m}^2 + \delta_3}{u_{n+1,m} - u_{n-1,m}} \right] \partial_{u_{n,m}}, \quad (18e)$$

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<sup>1</sup>Note that the equation  $D_3$  (14d) is invariant under the exchange  $n \leftrightarrow m$  so the symmetry  $X_m^{D_3}$  (18b) can be obtained from the symmetry  $X_n^{D_3}$  (18a) performing such exchange.

$$\begin{aligned} \hat{X}_m^{2D_4} = & \left[ \frac{\delta_2 F_{n+m}^{(+)} u_{n,m+1} u_{n,m-1} + \frac{1}{2} u_{n,m} (u_{n,m+1} + u_{n,m-1})}{u_{n,m+1} - u_{n,m-1}} + \right. \\ & \left. + \frac{\delta_2 F_{n+m}^{(-)} u_{n,m}^2 - \delta_1 \delta_3 F_n^{(+)}}{u_{n,m+1} - u_{n,m-1}} \right] \partial_{u_{n,m}}, \end{aligned} \quad (18f)$$

and no point symmetries. Again the fluxes of the symmetries (18) can be readily identified with some specific form of the non autonomous YdKN equations (3) and the explicit form of the coefficients are shown in Table 4.

### 3 Algebraic entropy for the non autonomous YdKN equation and its subcases.

In the previous section we saw that the fluxes of all the generalized three point symmetries of the  $H^4$  and  $H^6$  equations are eventually related either to the YdKN (1) or to the non autonomous YdKN equation (3).

It was remarked in the introduction that the non autonomous YdKN equation (3) passes the necessary condition for the integrability which is only an indication of the integrability of such class of equation. In this paper we have shown that they are symmetries of the  $H^4$  and  $H^6$  equations. Here we give a further evidence that the non-autonomous YdKN might be an integrable differential-difference equations based on the algebraic entropy test [3].

We recall briefly how to compute the algebraic entropy in the case of differential-difference equations of the form  $du_n/dt = f_n(u_{n+1}, u_n, u_{n-1})$  [7, 17]. First of all we assume that the equation is solvable for  $u_{n+1}$  uniquely. This is a condition on  $f_n$ . Then, starting from  $n = 0$ , we compute  $u_1$  by substituting the initial conditions:

$$\left\{ \frac{d^k u_{-1}}{dt^k}, \frac{d^k u_0}{dt^k} \right\}_{k=0}^{\infty}. \quad (19)$$

Knowing  $u_1$  we can then calculate  $u_2$  and so on. We define the degree of the iterate at the  $l$ -th step as the maximum between the degree of the numerator and of the denominator of  $u_l$  in the initial conditions (19). A great simplification in the explicit calculations is obtained if instead of a generic initial condition one parametrizes the curve of the initial condition rationally using the variable  $t$ :

$$u_{-1} = \frac{A_{-1}t + B_{-1}}{At + B}, \quad u_0 = \frac{A_0t + B_0}{At + B}, \quad (20)$$

and then compute the degrees in  $t$ . Calculating  $N$  iterates, for a sufficiently large positive integer  $N$ , and constructing the generating function one can calculate the algebraic entropy without calculating the entire sequence. For more details on how the method is implemented see [8].

We look for the sequence of degrees of the iterate map for the non autonomous YdKN equation (3) and its particular cases found in the previous section. We find for all the cases, except the symmetries of  ${}_r H_1^\varepsilon$ , the symmetries (11, 13) for  ${}_t H_1^\varepsilon$  and the symmetries of the  ${}_i D_2$  equations, the following values:

$$1, 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, 157 \dots \quad (21)$$

Eq.	$k$	$a$	$b_k$	$c_k$	$d$	$e_k$	$f$
$D_3$	$n$	0	0	$F_n^{(+)} F_m^{(+)}$	0	$\frac{1}{2} \left( F_n^{(+)} F_m^{(-)} + F_n^{(-)} F_m^{(-)} - F_n^{(+)} F_m^{(+)} \right)$	0
	$m$	0	0	$F_n^{(+)} F_m^{(+)}$	0	$\frac{1}{2} \left( F_n^{(-)} F_m^{(+)} + F_n^{(-)} F_m^{(-)} - F_n^{(+)} F_m^{(+)} \right)$	0
${}_1D_4$	$n$	0	0	$-\delta_1 \left( F_n^{(+)} F_m^{(+)} + F_n^{(+)} F_m^{(-)} \right)$	$-\frac{1}{2}$	0	$\delta_2 \delta_3 F_m^{(+)}$
	$m$	0	0	$\delta_2 \left( F_n^{(+)} F_m^{(+)} + F_n^{(-)} F_m^{(+)} \right)$	$\frac{1}{2}$	0	$-\delta_1 \delta_3 F_n^{(+)}$
${}_2D_4$	$n$	0	0	$-F_n^{(+)} F_m^{(+)} \delta_1 \delta_2$	$\frac{1}{2}$	0	$\delta_3$
	$m$	0	0	$\delta_2 \left( F_n^{(+)} F_m^{(+)} + F_n^{(-)} F_m^{(-)} \right)$	$\frac{1}{2}$	0	$-\delta_1 \delta_3 F_n^{(+)}$

Table 4: Identification of the coefficients of the symmetries (18) for  $D_3$ ,  ${}_1D_4$  and  ${}_2D_4$  with those of a non autonomous YdKN.

This sequence has the following generating function:

$$g(z) = \frac{1 - 2z + 3z^2}{(1 - z)^2}, \quad (22)$$

which gives the following quadratic fit for the sequence (21):

$$d_l = l(l - 1) + 1. \quad (23)$$

For the symmetry in the  $n$  direction (5a) of the equation  ${}_rH_1^\varepsilon$  we have the somehow different situation when the sequence growth is different according to the even or odd values of the  $m$  variable:

$$m = 2k \quad 1, 1, 3, 7, 10, 17, 23, 33, 42, 55, 67, 83, 98, 117 \dots \quad (24a)$$

$$m = 2k + 1 \quad 1, 1, 3, 4, 9, 13, 21, 28, 39, 49, 63, 76, 93, 109 \dots \quad (24b)$$

These sequences have the following generating functions and asymptotic fits:

$$\begin{aligned} m = 2k, \quad g(z) &= \frac{2z^5 - 3z^4 + 3z^3 + z^2 - z + 1}{(1 - z)^3(z + 1)}, \\ d_l &= \frac{3}{4}l^2 - l - \frac{5(-1)^l - 21}{8}, \end{aligned} \quad (25a)$$

$$\begin{aligned} m = 2k + 1, \quad g(z) &= \frac{(z^2 + z + 1)(2z^2 - 2z + 1)}{(1 - z)^3(z + 1)}, \\ d_l &= \frac{3}{4}l^2 - \frac{3}{2}l - \frac{5(-1)^l - 19}{8}. \end{aligned} \quad (25b)$$

The symmetry in the  $m$  direction (7b) of the equation  ${}_rH_1^\varepsilon$  has the same behaviour by exchanging  $m$  with  $n$  in formulae (24-25). The  $n$  directional symmetry of the equation  ${}_tH_1^\varepsilon$  has almost the same growth for  $m$  odd as (24b, 25b) however it is worthwhile to mention that the fit  $d_l = \frac{3}{4}l^2 - \frac{5}{4}l + (-1)^l \frac{l}{4} + \frac{(-1)^{l+15}}{8}$  presents a term  $l(-1)^l$ , new in this kind of results. For  $m$  even we have the same growth as (21). The  $m$  directional symmetry of the equation  ${}_tH_1^\varepsilon$  has the same growth as the even one of  ${}_tH_1^\varepsilon$  (24a, 25a). For the symmetries (17) of the  ${}_iD_2$  equations we have different growth according to the even or odd values of the  $m$  or  $n$  variables and similar sequences or slightly lower than in the case of equations  $H_1^\varepsilon$ , however always corresponding to a quadratic asymptotic fit.

This shows that the whole family of the non autonomous YdKN is integrable according to the algebraic entropy test.

For completeness let us just mention that the symmetries (15) of the  ${}_iD_2$  equations have a sequence growth of the same order than those considered above, i.e. quadratic growth and thus null entropy.

## 4 Conclusions

In this note we constructed the symmetries of the equations belonging to the Boll classification [5, 6] and showed that they are integrable (by the algebraic entropy test) and related to particular cases of the non autonomous YdKN equation (3) [15]. This was already known for the rhombic  $H^4$  equations [19]

and here we show the explicit identification of the symmetries obtained in that paper with the coefficients of the non autonomous YdKN equation.

We finally note that, as was proved in [12] for the YdKN (1), no equation belonging to the Boll classification has a symmetry which corresponds to the general non autonomous YdKN equation (3). In all the cases of the Boll classification one has  $a = b_k = 0$ .

In [18] it was shown that the  $Q_V$  equation introduced by Viallet, possessing the Klein symmetry,

$$\begin{aligned}
Q_V : \quad & p_4 + p_3 (u_{n,m} + u_{n,m+1} + u_{n+1,m} + u_{n+1,m+1}) + \\
& + p_{2,1} (u_{n,m} u_{n+1,m} + u_{n,m+1} u_{n+1,m+1}) + \\
& + p_{2,2} (u_{n,m} u_{n,m+1} + u_{n+1,m} u_{n+1,m+1}) + \\
& + p_{2,0} (u_{n,m} u_{n+1,m+1} + u_{n,m+1} u_{n+1,m}) + \\
& + p_1 (u_{n,m} u_{n,m+1} u_{n+1,m} + u_{n,m} u_{n+1,m} u_{n+1,m+1} + \\
& + u_{n,m} u_{n,m+1} u_{n+1,m+1} + u_{n,m+1} u_{n+1,m} u_{n+1,m+1}) + \\
& + p_0 u_{n,m} u_{n,m+1} u_{n+1,m} u_{n+1,m+1} = 0,
\end{aligned} \tag{26}$$

admits a symmetry of the form of the YdKN:

$$\hat{X}_n^V = \frac{h}{u_{n+1,m} - u_{n-1,m}} - \frac{1}{2} \partial_{u_{n+1,m}} h. \tag{27}$$

where:

$$\begin{aligned}
h(u_{n,m}, u_{n+1,m}; p_1, p_{2,i}, p_3, p_4) = & Q_V \partial_{u_{n,m+1}} \partial_{u_{n+1,m+1}} Q_V + \\
& - (\partial_{u_{n,m+1}} Q_V) (\partial_{u_{n+1,m+1}} Q_V).
\end{aligned} \tag{28}$$

The connection formulae between the coefficient of  $Q_V$  and the YdKN (1) is:

$$\begin{aligned}
a &= p_1^2 - p_{2,1} p_0, & b &= \frac{1}{2} [p_1 (p_{2,0} + p_{2,2} - p_{2,1}) - p_3 p_0], \\
c &= p_{2,0} p_{2,2} - p_3 p_1, & d &= \frac{1}{2} [p_{2,2}^2 - p_{2,1}^2 + p_{2,0}^2 - p_0 p_4], \\
e &= \frac{1}{2} [p_3 (p_{2,2} - p_{2,1} + p_{2,0}) - p_1 p_4], & f &= p_3^2 - p_{2,1} p_4.
\end{aligned} \tag{29}$$

This is a set of coupled nonlinear algebraic equations between the 7 parameters  $p_i$  of  $Q_V$  and the 6 ones  $(a, \dots, f)$  of the YdKN. Eq. (29) tells us that the YdKN with coefficients given by (29) is a three point generalized symmetry of  $Q_V$ . If a solution of (29) exists, i.e. one is able to express the  $p_i$  in term of  $(a, \dots, f)$ , then (a reparametrization of)  $Q_V$  turns out to be a Bäcklund transformation of the YdKN [11].

From the results obtained in this paper one is lead to conjecture a non autonomous generalization of the  $Q_V$  equation. We have many possible ways of proposing such a generalization. A first possibility is to generalize the original Klein symmetry:

$$\begin{aligned}
& Q(u_{n+1,m}, u_{n,m}, u_{n+1,m+1}, u_{n,m+1}; (-1)^n, (-1)^m) = \\
& \tau Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; (-1)^n, (-1)^m), \\
& Q(u_{n,m+1}, u_{n+1,m+1}, u_{n,m}, u_{n,m+1}; (-1)^n, -(-1)^m) =
\end{aligned} \tag{30}$$



$$\tau' Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; (-1)^n, (-1)^m),$$

where  $(\tau, \tau') = \pm 1$  and  $Q(x, u, y, z; (-1)^n, (-1)^m)$  is a multilinear function of its arguments with nonautonomous coefficients in the form of 2-periodic functions in  $n$  and  $m$ , i. e. of the form  $\alpha + \beta(-1)^n + \gamma(-1)^m + \delta(-1)^{n+m}$ , with  $\alpha, \beta, \gamma$  and  $\delta$  constants. This discrete symmetry is shared by all of the Boll systems and in the autonomous case reduces to the usual Klein symmetry.

A second possibility is to ask the function  $Q(x, u, y, z; (-1)^n, (-1)^m)$  to respect a strict Klein symmetry just as in (26). Choosing the coefficients for example as

$$p_0 = 1 + (-1)^n, \quad p_1 = (-1)^n, \quad p_{2,1} = -1 + (-1)^n, \quad p_{2,2} = (-1)^n,$$

$$p_{2,0} = 1 + 2(-1)^n, \quad p_3 = 1 + (-1)^n, \quad p_4 = 4 + 2(-1)^n,$$

(29) provide a non autonomous YdKN. In this case, performing the algebraic entropy test the equation turns out to be integrable. Its generalized symmetries, however, are not necessarily in the form of a non autonomous YdKN equation. A different non autonomous choice of the coefficients of (26), such that (29) is satisfied for the coefficients of the non autonomous YdKN, gives, by the algebraic entropy test, a non integrable equation.

The proof of the existence of a non autonomous generalization of  $Q_V$  together with the derivation of an effective Bäcklund transformation and Lax pair for the YdKN and its non autonomous counterpart is work in progress.

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## References

- [1] V. E. Adler, A. I. Bobenko, and Y. B. Suris. Classification of integrable equations on quad-graphs. the consistency approach. *Comm. Math. Phys.*, **233**, 513–543, 2003.
- [2] V. E. Adler, A. I. Bobenko, and Y. B. Suris. Discrete nonlinear hyperbolic equations. classification of integrable cases. *Funct. Anal. Appl.*, **43**, 3–17, 2009.
- [3] M. Bellon and C-M. Viallet, Algebraic Entropy, *Comm. Math. Phys.*, **204**, 425–437, 1999.
- [4] R. Boll. Classification of 3D consistent quad-equations. *J. Nonl. Math. Phys.*, **18**, 337–365, 2011.
- [5] R. Boll, Corrigendum Classification of 3D consistent quad-equations, *J. Nonl. Math. Phys.* **19**, 1292001 (3 pp), 2012.

- [6] R. Boll, Classification and Lagrangian structure of 3D consistent quad-equations, Ph. D. dissertation, 2012.
- [7] D.K. Demskoy and C.-M. Viallet, Algebraic entropy for semi-discrete equations. *J. Phys. A: Math. Theor.* **45**, 352001 (10 pp), 2012.
- [8] G. Gubbiotti Ph. D. Dissertation to be submitted in 2016.
- [9] G. Gubbiotti, C. Scimiterna and D. Levi, On quad equations consistent on the cube, submitted to *J. Phys. A: Math. Theor.* .
- [10] G. Gubbiotti, C. Scimiterna and D. Levi, Linearizability and fake Lax pair for a consistent around the cube nonlinear non-autonomous quad-graph equation, submitted to *Teoreticheskaya i Matematicheskaya Fizika*.
- [11] D. Levi, Nonlinear differential-difference equations as Bäcklund transformations, *J. Phys. A: Math. Gen.* **14**, 1083–1098, 1981.
- [12] D. Levi, M. Petrera, C. Scimiterna, and R. I. Yamilov. On Miura transformations and Volterra-type equations associated with the Adler–Bobenko–Suris equations. *SIGMA*, **4**, 077 (14pp), 2008.
- [13] D. Levi, P. Winternitz and R. I. Yamilov, Symmetries of the continuous and discrete Krichever–Novikov equation, *SIGMA*, **7**, 097 (21 pp), 2011.
- [14] D. Levi and R. I. Yamilov, Classification Of Evolutionary Equations On The Lattice I. The General Theory, arXiv:solv-int/9511006 v1 16 Nov 1995.
- [15] D. Levi and R. I. Yamilov. Conditions for the existence of higher symmetries of the evolutionary equations on the lattice. *J. Math. Phys.*, **38**, 6648–6674, 1997.
- [16] O. G. Rasin and P. E. Hydon. Symmetries of integrable difference equations on the quad-graph. *Stud. Appl. Math.*, **119**, 253–269, 2007.
- [17] C. M. Viallet, Algebraic entropy for differential–delay equations arXiv:1408.6161, 2014.
- [18] P. D. Xenitidis, Integrability and symmetries of difference equations: the Adler-Bobenko-Suris case, Proc. 4<sup>th</sup> Workshop Group Analysis of Differential Equations and Integrable Systems, (2009) 226–242 (arXiv:0902.3954).
- [19] P. D. Xenitidis and V. G. Papageorgiou. Symmetries and integrability of discrete equations defined on a black and white lattice. *J. Phys. A: Math. Theor.*, **42**, 454025 (13 pp), 2009.
- [20] R. I. Yamilov Classification of discrete evolution equations *Uspekhi Mat. Nauk* **38**, 155–156, 1983 (in Russian).
- [21] R. I. Yamilov, *Discrete equations of the form  $\dot{u}_n = F(u_{n-1}, u_n, u_{n+1})$   $n \in \mathbb{Z}$  with an infinite number of local conservation laws* Dissertation for Candidate of Science (Ufa: Soviet Academy of Sciences) 1984 (in Russian).
- [22] R. I. Yamilov, Symmetries as integrability criteria for differential difference equations, *J. Phys. A: Math. Gen.* **39**, R541–R623, 2006.